1 Find all vector spaces that have exactly one basis.

- First think about $\mathbb{R}$ (It is very easy to think through the example). Then trival vector vectorspace has only one basis $\{0\}$.

  because if there is a non-zero γ in a basis, then then for non-unit element $c \in \mathbb{F}$, we get $c\gamma$ also basis for V.
- $\mathbb{C}$ is also have same thing. $\to$ $\{0\}$.
- Now let's consider finite feild

Result: If $\{\gamma_1, \ldots, \gamma_n\}$ is a basis for a vector space V. then $\{\gamma_1+\gamma_2, \gamma_2+\gamma_3, \ldots, \gamma_n\}$ is a basis for V (proof: Ex-2B/07)

Result 2: If $\gamma \in \text{basis}(V)$ then $c\gamma \in \text{basis}(V)$ for all $0 \neq c \in \mathbb{F}$.

V must be a vector space with dimensin one on a feild isomorphic to $\mathbb{Z}_2$.

Thus, $V = \{0, \gamma\}$ or $V = \{0\}$ are vector space that have only one basis.

2 Verify all assertions in Example 2.27.

(a) The list $(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ is a basis of $\mathbf{F}^n$, called the *standard basis* of $\mathbf{F}^n$.

Claim: $(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,\ldots,1)$ is linearly independent.

Suppose that there exist $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that

$$a_1(1,0,\ldots,0)+a_2(0,1,\ldots,0)+\cdots+a_n(0,\ldots,1)=(0,0,\ldots,0)$$
$$(a_1+\cdots+a_n)=(0,\ldots,0)$$

This implies, $a_1=a_2=\cdots=a_n=0$

Hence, $(1,\ldots,0), \ldots, (0,\ldots,1)$ are linearly independent.

Claim2: $\operatorname{Span}((1,\ldots,0), \ldots, (0,\ldots,1))=\mathbb{F}^n$

Let $(b_1,b_2,\ldots,b_n)\in\mathbb{F}^n$. Then,

$$(b_1,b_2,\ldots,b_n)=b_1(1,0\ldots,0)+b_2(0,1,\ldots,0)+\cdots+b_n(0,\ldots,1)$$

Thus, $\operatorname{Span}((1,\ldots,0), \ldots, (0,\ldots,1))=\mathbb{F}^n$

(b) The list $(1,2)$, $(3,5)$ is a basis of $\mathbf{F}^2$. Note that this list has length two, which is the same as the length of the standard basis of $\mathbf{F}^2$. In the next section, we will see that this is not a coincidence.

Claim: $(1,2),(3,5)$ is linearly independent.

Suppose that there exist $a_1,a_2 \in \mathbb{F}$ such that

$$a_1(1,2)+a_2(3,5)=(0,0)$$
$$(a_1+3a_2, 2a_1+5a_2)=(0,0)$$

This implies,

$$a_1+3a_2=0 \quad \text{——} \ (1)$$
$$2a_1+5a_2=0 \quad \text{——} \ (2)$$

$2\times(1)-(2)$, $a_2=0$. Then by (1) $a_1+3a_2=a_1+0=0$
$a_1=0$.

Thus. $(1,2),(3,5)$ are linearly independent.

Claim: $\operatorname{Span}((1,2),(3,5))=\mathbb{F}^2$.

Let $(b_1,b_2)\in\mathbb{F}^2$. Then,

$$(b_1,b_2)=(3b_2-2b_1)(1,2)+(b_1-b_2)(3,5)$$

Thus, $\mathbb{F}^2\subseteq \operatorname{span}((1,2),(3,5))$. It is trivial that $\operatorname{span}((1,2),(3,5))\subseteq\mathbb{F}^2$. Thus
$\operatorname{span}((1,2),(3,5))=\mathbb{F}^2$

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

a) Let $V_1 = (3, 1, 0, 0, 0)$, $V_2 = (0, 0, 7, 1, 0)$.
$V_3 = (0, 0, 0, 0, 1)$.

Claim1: V_1, V_2, V_3 are linearly independent

Suppose that there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1 V_1 + a_2 V_2 + a_3 V_3 = 0$$

$$a_1(3, 1, 0, 0, 0) + a_2(0, 0, 7, 1, 0) + a_3(0, 0, 0, 0, 1) = (0, 0, 0, 0, 0)$$

$$(3a_1, a_1, 7a_2, a_2, a_3) = (0, \ldots, 0)$$

Thus, $a_1 = a_2 = a_3 = 0$

claim2: $\text{span}(V_1, V_2, V_3) = U$

Let $(b_1, b_2, b_3, b_4, b_5) \in U$. Then $b_1 = 3b_2$ & $b_3 = 7b_4$

$$(b_1, b_2, b_3, b_4, b_5) = (3b_2, b_2, b_4, 7b_4, b_5)$$

$$= b_2(3, 1, 0, 0, 0) + b_4(0, 0, 1, 7, 0) + b_5(0, \ldots, 1)$$

$$= b_2 V_1 + b_4 V_2 + b_5 V_5$$

Thus s, V_1, V_2, V_3 span U. Thus. V_1, V_2, V_3 is basis for U

b) $V_4 = (1,0,0,0,0)$ and $V_5 = (0,0,1,0,0)$

claim3: $V_1, V_2, \dots, V_5$ are linearly independent

Suppose that there exist, $c_1, \dots, c_5 \in \mathbb{R}$ such that

$$c_1 V_1 + \cdots + c_2 V_2 = 0$$

$$c_1(3,1,0,0,0) + c_2(0,0,7,1,0) + c_3(0,0,0,0,1) + c_4(1,0,0,0,0) + c_5(0,0,1,0,0) = (0,0,0,0,0)$$

Thus, $(3c_1 + c_4, c_1, 7c_2 + c_5, c_2, c_3) = (0,0,0,0,0)$

$c_3 = 0$	$3c_1 + c_4 = 0$	$7c_2 + c_5 = 0$
$c_2 = 0$	$c_4 = 0$	$c_5 = 0$
$c_1 = 0$.		

Thus, $V_1, \dots, V_5$ are linearly independent

claim4: $(V_1, \dots, V_5)$ spans $\mathbb{R}^5$

Let $(d_1, \dots, d_5) \in \mathbb{R}^5$. Then,

$$(d_1, d_2, d_3, d_4, d_5) = d_1(3,1,0,0,0) + d_2(0,0,7,1,0) + d_3(0,0,0,0,1) + d_4(1,0,0,0,0) + d_5(0,0,1,0,0)$$

$$= d_1 V_1 + d_2 V_2 + d_3 V_3 + d_4 V_4 + d_5 V_5$$

Thus, $\mathbb{R}^5$ span by $V_1, \dots, V_5$. Thus, $V_1, \dots, V_5$ are basis for $\mathbb{R}^5$

c) $W = \text{span}((1,0,0,0),(0,0,1,0,0))$

Then $V+W = \mathbb{R}^5$ (by part b))

Furthur $V \cap W = \{0\}$.

Thus $V \oplus W = \mathbb{R}^5$.

4 (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

claim1: $(1,6,0,0,0)$, $(0,0,2,-1,0)$ and $(0,0,3,0,-1)$ is linearly independent

Suppose that $a, b, c \in \mathbb{F}$ such that

$a(1,6,0,0,0) + b(0,0,2,-1,0) + c(0,0,3,0,-1) = 0$

$(a, 6a, 2b+3c, -b, -c) = (0,0,0,0,0)$

Then $a = b = c = 0$

Thus $(1,6,0,0,0)$, $(0,0,2,-1,0)$, $(0,0,3,0,-1)$ are linearly independendt ——— $\circledast$

~~Let~~ Claim2: $(1,6,0,0,0)$, $(0,0,2,-1,0)$, $(0,0,3,0,-1)$ is spans U

Let $v = (z_1, z_2, z_3, z_4, z_5) \in U$

Then $6z_1 = z_2$

$z_3 + 2z_4 + 3z_5 = 0$

$z_3 = -2z_4 - 3z_5$

$-2z_4 - 3z_5)$

$$v = (z_1, z_2, z_3, z_4, z_5)$$

$$= (z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5)$$

$$= z_1(1, 6, 0, 0, 0)$$

$$-z_4(0, 0, 2, -1, 0)$$

$$-z_5(0, 0, 3, 0, -1)$$

Thus $(1,6,0,0,0)$, $(0,0,2,-1,0)$, $(0,0,3,0,-1)$ spans U.

Therefore $(1,6,0,0,0)$, $(0,0,2,-1,0)$ and $(0,0,3,0,-1)$ are basis for U.

b) $(1,6,0,0,0)$, $(0,0,2,-1,0)$, $(0,0,3,0,-1)$, $(0,1,0,0)$, $(0,0,1,0,0)$ is a basis for $\mathbb{C}^5$.

Claim 1: $(1,6,0,0,0),(0,0,2,-1,0),(0,0,3,0,-1)$
$(0,1,0,0,0),(0,0,1,0,0)$ is
linearly independent

Suppose that there exist $a,b,c,d,e \in \mathbb{F}$
such that

$$a(1,6,0,0,0)+b(0,0,2,-1,0)+c(0,0,3,0,-1)$$
$$+d(0,1,0,0,0)+e(0,0,1,0,0)$$
$$=(a,6a+d,2b+e+3c,-b,-c)=(0,0,0,0,0)$$

Then $a=b=c=0$

$6a+d=0 \Rightarrow d=0$

$2b+e=0 \Rightarrow e=0$

Therefore given vectors are linearly independent.

Claim2: $(1,6,0,0,0), (0,0,2,-1,0), (0,0,3,0,-1)$
$(0,0,1,0,0), (0,1,0,0,0)$ spans $\cancel{\mathbb{R}^5}\,\mathbb{C}^5$

Let $\gamma = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$

$$\gamma = z_1(1,6,0,0,0) + (z_2 - 6z_1)(0,1,0,0,0)$$
$$+ (z_3 + 2z_4 + 3z_5)(0,0,1,0,0)$$
$$+ (-1)z_5(0,0,3,0,-1)$$
$$+ (-1)(z_4)(0,0,2,(-1),0)$$

Thus given vectors are spans $\mathbb{C}^5$.

of the
in this

c) $W = \text{span}((0,1,0,0,0), (0,0,1,0,0))$

Then $U+W = \mathbb{C}^5$ (by part b)

Furthur $U \cap W = \{0\}$.

Then $U \oplus W = \mathbb{C}^5$.

5 Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Suppose that V is finitely dimensional and U, W are subspace of V, such that $V = U + W$.

Let $(u_1, u_2, \ldots, u_n)$ be a basis for U and $(w_1, \ldots, w_m)$ be a basis for W

& Let $\gamma \in V = U + W$.

$$\gamma = a_1 u_1 + \cdots + a_n u_n + b_1 w_1 + \cdots + b_m w_m$$

Thus, $u_1, \ldots, u_n, w_1, \ldots, w_m$ spans V

Recall 2.30 from the book.

Every spanning list in a vector space can be reduced to basis of the vector space

Therefore $u_1, \ldots, u_n, w_1, \ldots, w_m$ can be reduced in to a basis.

Therefore there exista basis of V consisting of vectors in $U \cup W$.

6 Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbf{F})$.

Consider $P_0 = 1 + x^3 \rightarrow \deg(P_0) = 3$

$P_1 = x + x^3 \rightarrow \deg(P_1) = 3$

$P_2 = x^2 + x^3 \rightarrow \deg(P_2) = 3$

$P_3 = x^3 \rightarrow \deg(P_3) = 3.$

Supose that $a, b, c, d \in \mathbb{F}$ suchthat

$$a(1+x^3) + b(x+x^3) + c(x^2+x^3) + d\,x^3 = 0$$

$$a + bx + cx^2 + (a+b+c+d)\,x^3 = 0$$

We knowt $1, x, x^2, x^3$ are linearly independent

$$a = b = c = d = 0$$

Thus P_0, P_1, P_2, P_3 are linearnly independent.

Let $P \in \mathcal{P}_3(\mathbb{F})$. Since $1, x, x^2, x^3$ is the standered basis, there exist $l, m, n, r \in \mathbb{F}$ such that,

$$P = l + mx + nx^2 + rx^3$$

$$1 = (1+x^3) - x^3 = P_0 - P_3$$

$$x = (x+x^3) - x^3 = P_1 - P_3$$

$$x^2 = (x^2+x^3) - x^3 = P_2 - P_3$$

Thus $P = l(P_0 - P_3) + m(P_1 - P_3) + n(P_2 - P_3) + r$

$$= lP_0 + mP_1 + nP_2 - (l+m+n-r)P_3$$

Thus. $\text{span}(P_0, P_1, P_2, P_3) = \mathcal{P}_3(\mathbb{F})$.

Therefore P_0, P_1, P_2, P_3 is a basis of $\mathcal{P}(\mathbb{F})$ but none of have degree 2.